From now on, when I say "Graph" I mean an undirected graph, unless stated otherwise.

(usually, there won't be loops, but often that's OK)

Given a Graph (V, E), we make a matrix A (called the <u>adjacency matrix</u>) - If $n = |V|$, we have an $n \times n$ matrix

$$A = \begin{pmatrix} - & - & - & - & - \\ - & - & - & & \\ - & - & & & \end{pmatrix}$$

columns

$$\begin{array}{c} \text{row} \end{array} \quad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \\ A_{21} & A_{22} & A_{23} & \\ & & & \end{pmatrix}$$

$A_{23} = 2^{nd}$ row, 3^{rd} column of the matrix.

Example

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 1 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \longleftarrow 5 \times 5 \text{ matrix}$$

rows columns

$A_{31} = 2$ $\quad$ $A_{35} = 4$

In general $A_{ij} = i^{th}$ row, j^{th} column entry of the matrix A.

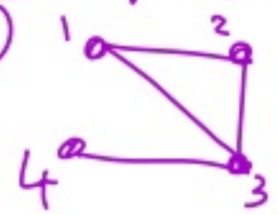
Adjacency matrix A of a graph $G = (V, E)$ is defined by

$$A_{ij} = \#\text{ of edges from } i \text{ to } j. \quad \left(\begin{array}{l} |V| = n \\ A \text{ is an } n \times n \text{ matrix} \end{array}\right)$$

(for directed graphs — the order is important. For undirected graphs, $A_{ij} = A_{ji}$.)

Examples

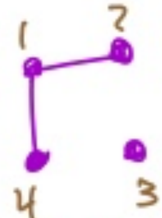
①



②

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

③



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

④

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 \end{array}$$

Adjacency matrices:

①

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array} \quad \text{or} \quad \begin{array}{c|cccc} & 2 & 1 & 3 & 4 \\ \hline 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

$$\text{or} \quad \begin{array}{c|cccc} & 4 & 2 & 3 & 1 \\ \hline 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{array}$$

Given an adjacency matrix, we can make a graph:

$$\underset{\text{directed.}}{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \Longrightarrow$$

How do we tell if 2 graphs are the same (isomorphic)?

We say graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection $f: V_1 \to V_2$ such that if $a, b \in V_1$ then a is connected to b in G_1 if and only if f(a) is connected to f(b) in G_2.

bijection = 1-1 and onto map

$\{a,b\} \in E_1$
$\Leftrightarrow \{f(a), f(b)\} \in E_2$

Example Are these Graphs isomorphic?

Yes $G_1 \cong G_2$ (G_1 is isomorphic to G_2)

$f: G_1 \to G_2$

$f(a) = 6$
$f(b) = 2$
$f(c) = 3$
$f(d) = 4$ $f(e) = 5$ $F(f) = 1$

$E_1 = \{\{a,b\}, \{a,d\}, \{b,c\}, \{b,e\}, \{c,d\}, \{c,f\}\}$

$E_2 = \{\{6,2\}, \{6,4\}, \{2,3\}, \{2,5\}, \{3,4\}, \{3,1\}\}$

Harder: Proving graphs are not isomorphic.